

# Two impurities in a Bose-Einstein condensate: from Yukawa to Efimov attracted polarons

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The properties of two particles immersed in a Bose-Einstein condensate are investigated using a variational approach. These properties can be interpreted in terms of a boson-mediated interaction. For a weak particle-boson attractive interaction, the two particles form two free attractive polarons that interact through a weak boson-mediated Yukawa attraction. For sufficiently strong interactions, the boson-mediated interaction turns into an Efimov attraction leading to the binding of the two particles and one boson into Efimov trimers. It is found that these in-medium Efimov trimers exist for weaker interactions than in vacuum.

A particle interacting with a surrounding medium can form a polaron, i.e. it becomes dressed by a cloud of excitations of the medium that alters its properties. This general concept, introduced by Landau and Pekar [1] to describe electrons coupled to the vibrations of a lattice in solids, has proved useful to understand a variety of physical systems such as semi-conductors and superconductors [2]. In the last few years, polarons with arbitrarily strong interactions with the medium could be investigated experimentally using ultra-cold atoms [3–6]. These experiments have realised Fermi polarons (impurities embedded in a Fermi sea) by mixing different kinds of fermionic atoms and tuning their interaction by a Feshbach resonance. Recently, two experimental works [7, 8] have reported the observation of Bose polarons (impurities embedded in a Bose gas) by using bosonic ultra-cold atoms. While the properties of a single Bose polaron are interesting and theoretically challenging [9–23], it is also of fundamental interest to understand the interaction between Bose polarons induced by their medium. Such an induced interaction occurs in mixtures of bosonic and fermionic helium liquids [24]. A similar phenomenon appears in high-energy physics, where the nuclear force is mediated by mesons [25]. These situations are known to lead to a Yukawa potential between the particles, in the perturbative regime of weak interaction between the particles and the massive bosons. The non-perturbative regime, on the other hand, has been less explored.

At the few-body level, it is known that for sufficiently strong interactions, an effective three-body force called the Efimov attraction can bind three particles into one of infinitely many three-body bound states, known as Efimov trimers [26–28]. The Efimov attraction scales as the inverse square of the distance between particles, conferring discrete scale invariance to the system. Efimov trimers and their singular properties have been observed in ultra-cold atom experiments in the last few years [29], triggering the question of the influence of the surrounding medium on these trimer states. A theoretical study [23] treating the problem of a single impurity strongly interacting with a condensate has shown how the resulting polaron turns into an Efimov trimer due to the Efimov

attraction between the impurity and two bosons - a similar effect was found in [30], as well as for an impurity in a two-component Fermi superfluid [31, 32]. Reference [23] found that the in-medium Efimov trimer is stabilised by the surrounding condensate. Another study [33] has looked into the problem of two heavy impurities in a Bose-Einstein condensate. In this case too, the two impurities are expected to form Efimov trimers with one of the bosons for sufficiently strong attraction. The study suggested that the Efimov trimers are weakened by the condensate. However, the theory could not completely describe the interaction at large distance between the two impurities induced by the Bose-Einstein condensate. The precise effect of a surrounding Bose-Einstein condensate on Efimov trimers and the mediated interaction thus remain to be clarified.

Motivated by these theoretical questions and the recent experiments with ultra-cold atoms, this work presents a minimal description of two impurities in a Bose-Einstein condensate that covers both the perturbative regime of weakly Yukawa-attracted polarons and the non-perturbative regime corresponding to a bound Efimov trimer immersed in a Bose-Einstein condensate. This description is based on the method of Refs. [23, 34], which uses a variational wave function for the impurities and the excitations of the medium from its ground state. Here, the excitations are the Bogoliubov quasiparticles of the condensate. In the following, only a single excitation will be considered, which is the minimal requirement to reproduce the expected Efimov three-body physics. First, the mediated interaction between the two impurities will be derived, and then the energy spectrum of the system will be presented and discussed.

The Bose-Einstein condensate is assumed to be homogeneous and weakly self-interacting via a pairwise interaction  $U_B$  between bosons, whereas the interaction  $U$  between an impurity and a boson may be arbitrarily strong. No direct interaction between the impurities is considered. The impurities are also assumed to be identical bosons. The Hamiltonian thus reads in second-

quantisation:

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{p}} U_B(\mathbf{p}) b_{\mathbf{k}'}^{\dagger} b_{\mathbf{k}+\mathbf{p}}^{\dagger} b_{\mathbf{k}} b_{\mathbf{k}'} \quad (1)$$

$$+ \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{p}} U(\mathbf{p}) b_{\mathbf{k}'}^{\dagger} b_{\mathbf{k}+\mathbf{p}}^{\dagger} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}'} b_{\mathbf{k}}$$

where  $V$  is the system's volume,  $\epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}$  and  $b_{\mathbf{k}}$  are the kinetic energy and annihilation operator for a boson with momentum  $\mathbf{k}$ , and  $\varepsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2M}$  and  $c_{\mathbf{k}}$  are the kinetic energy and annihilation operator for an impurity with momentum  $\mathbf{k}$ . Since the bosons are weakly interacting, the first line of Eq. (1) can be approximately diagonalised as  $\mathcal{E}_0 + \sum_{\mathbf{k}} E_{\mathbf{k}} \beta_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}}$ , by setting  $b_0 = \sqrt{N_0}$  and using for  $\mathbf{k} \neq 0$  the Bogoliubov transformation

$$b_{\mathbf{k}} = u_{\mathbf{k}} \beta_{\mathbf{k}} - v_{\mathbf{k}} \beta_{-\mathbf{k}}^{\dagger}, \quad (2)$$

where the operator  $\beta_{\mathbf{k}}$  annihilates a quasi-particle with momentum  $\mathbf{k}$ ,  $u_{\mathbf{k}}^2 = \frac{1}{2} \left( \frac{\epsilon_{\mathbf{k}} + n_0 U_B(0)}{E_{\mathbf{k}}} + 1 \right)$ ,  $v_{\mathbf{k}}^2 = \frac{1}{2} \left( \frac{\epsilon_{\mathbf{k}} + n_0 U_B(0)}{E_{\mathbf{k}}} - 1 \right)$ , and  $E_{\mathbf{k}}^2 = \epsilon_{\mathbf{k}} (\epsilon_{\mathbf{k}} + 2n_0 U_B(0))$ . For convenience, the origin of energy is set to the condensate ground-state energy  $\mathcal{E}_0$ .

The total wave function  $|\Psi\rangle$  of the system is assumed to be a superposition of two impurities on top of the Bose-Einstein condensate ground state  $|\Phi\rangle$ , and two impurities plus one excitation on top of  $|\Phi\rangle$ ,

$$|\Psi\rangle = \left( \sum_{\mathbf{q}} \alpha_{\mathbf{q}} c_{\mathbf{q}}^{\dagger} c_{-\mathbf{q}}^{\dagger} + \sum_{\mathbf{q}, \mathbf{q}'} \alpha_{\mathbf{q}, \mathbf{q}'} c_{\mathbf{q}}^{\dagger} c_{\mathbf{q}'}^{\dagger} \beta_{-\mathbf{q}-\mathbf{q}'}^{\dagger} \right) |\Phi\rangle. \quad (3)$$

Applying the variational principle  $\langle \delta\Psi | H - E | \Psi \rangle = 0$  to the Hamiltonian of Eq. (1) with the ansatz of Eq. (3), where  $\alpha_{\mathbf{q}}$  and  $\alpha_{\mathbf{q}, \mathbf{q}'}$  are varied independently, gives a set of two coupled equations:

$$(2\varepsilon_{\mathbf{q}} + 2nU(0) - E) \alpha_{\mathbf{q}} + \frac{\sqrt{N_0}}{V} \sum_{\mathbf{k}} U(\mathbf{k}) (u_{\mathbf{k}} - v_{\mathbf{k}}) (\alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}, -\mathbf{q}}) = 0, \quad (4)$$

$$(E_{\mathbf{k}} + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_{\mathbf{q}} + 2nU(0) - E) \alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}} + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} + \mathbf{k}) (u_{\mathbf{k}} u_{\mathbf{p}} + v_{\mathbf{k}} v_{\mathbf{p}}) (\alpha_{\mathbf{q}, -\mathbf{q}-\mathbf{p}} + \alpha_{\mathbf{q}-\mathbf{p}-\mathbf{k}, \mathbf{k}-\mathbf{q}}) + \frac{\sqrt{N_0}}{V} U(\mathbf{k}) (u_{\mathbf{k}} - v_{\mathbf{k}}) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0, \quad (5)$$

where  $n_0 = N_0/V$  is the condensate density and  $n = n_0 + \frac{1}{V} \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \approx n_0 (1 + \frac{8}{3\sqrt{\pi}} \sqrt{n_0 a_B})$  is the total density of bosons. Here,  $a_B = \frac{m}{4\pi\hbar^2} U_B(0)$  is the boson scattering length in the Born approximation.

Let us first consider a weak interaction  $U$ , i.e. that can be treated perturbatively. This imposes that the Born

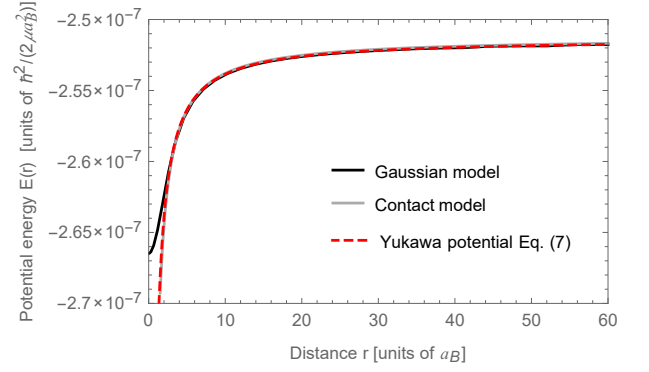


Figure 1. Mediated interaction  $E(r)$  between the two impurities for a boson-impurity scattering length  $a = -0.1a_B$ , with a condensate density  $n_0 = 10^{-7} a_B^{-3}$ , and a mass ratio  $M/m = 30$ . The solid curve corresponds to a Gaussian boson-impurity potential  $U(k) = U(0)e^{-\frac{1}{2}(\lambda k)^2}$  where  $U(0) = -0.0946 \times 4\pi\hbar^2\lambda/(2\mu)$  and  $\lambda = a_B$ . The dashed curve represents the Yukawa potential Eq. (7), with  $E(\infty) = 2 \times \frac{4\pi\hbar^2}{2\mu} n a (1 + \sqrt{2}a/\xi)$ . The solid grey curve corresponds to a contact model for the boson-impurity interaction.

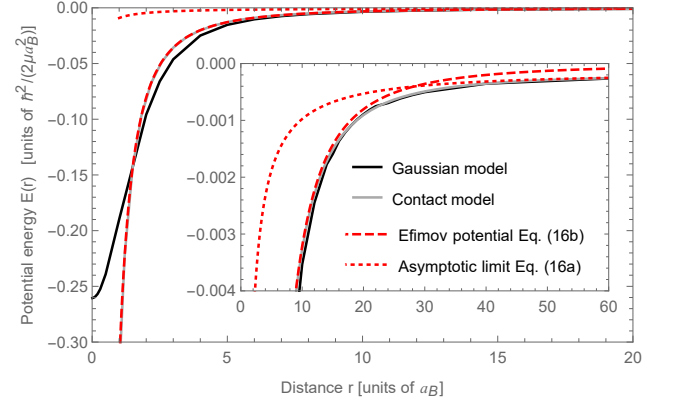


Figure 2. Same as Fig. 1, with  $a \approx \infty$ . The solid black curve corresponds to a Gaussian boson-impurity potential  $U(k) = U(0)e^{-\frac{1}{2}(\lambda k)^2}$  with  $U(0) = -1.6819 \times 4\pi\hbar^2\lambda/(2\mu)$ . The dashed and the dotted curves represent the Efimov potential and the asymptotic limit of Eq. (16). The solid grey curve corresponds to a contact model for the boson-impurity interaction.

expansion of the scattering length  $a = a_0 + a_1 + \dots$  converges rapidly, and  $a_0 = \frac{2\mu}{4\pi\hbar^2} U(0)$  is much larger than  $a_1 = -\frac{2\mu}{4\pi\hbar^2} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}}}$ , where  $\mu = (\frac{1}{M} + \frac{1}{m})^{-1}$  is the boson-impurity reduced mass. In this case, one can neglect the sum in Eq. (5), as it contributes to higher orders in  $U$ . Let us now fix the distance  $r$  between the two impurities, thus setting the kinetic energy  $\varepsilon_{\mathbf{k}}$  to zero, and perform a Fourier transform with respect to  $\mathbf{q}$ , the conjugate momentum of  $\mathbf{r}$ . Eliminating the second equation

into the first, one obtains

$$E' = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2 (u_k - v_k)^2}{E_k - E'} (1 + e^{i\mathbf{k} \cdot \mathbf{r}}), \quad (6)$$

where  $E' = E - 2nU(0)$ . The solution  $E(r)$  of this equation as a function of  $r$  gives the effective potential between the two impurities in the Born-Oppenheimer approximation. Equation (6) shows that it decays as a Yukawa potential (see appendix A),

$$E(r) \xrightarrow{r \rightarrow \infty} E(\infty) - \frac{8\pi\hbar^2 n_0}{2\mu^2/m} a_0^2 \frac{1}{r} e^{-\sqrt{2}r/\xi}, \quad (7)$$

where  $\xi = (8\pi n_0 a_B)^{-1/2}$  is the condensate coherence length, and  $E(\infty) \approx \frac{8\pi\hbar^2}{2\mu} \left( na_0 + \frac{m}{\mu} n_0 (a_1 + \sqrt{2}a_0^2/\xi) \right)$  is the asymptotic energy of the separated impurities, which is essentially twice the mean-field energy  $E_{MF} = \frac{4\pi\hbar^2}{2\mu} na$  of a single impurity [22].

If, on the other hand, the interaction  $U$  is not treated as a perturbation, one finds the following equation for  $E(r)$  (see appendix B):

$$E'(r) = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} U(\mathbf{k}) (u_k - v_k) A_{\mathbf{k}}(\mathbf{r}), \quad (8)$$

where  $A_{\mathbf{k}}(\mathbf{r})$  is obtained by inverting the equation

$$(1 + e^{i\mathbf{k} \cdot \mathbf{r}}) \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{k} - \mathbf{p}) (u_k u_p + v_k v_p) A_{\mathbf{p}}(\mathbf{r}) + (E_{\mathbf{k}} - E') A_{\mathbf{k}}(\mathbf{r}) = U(\mathbf{k}) (u_k - v_k) (1 + e^{i\mathbf{k} \cdot \mathbf{r}}). \quad (9)$$

Figure 1 shows the mediated potential  $E(r)$  calculated from Eqs. (8-9) for a weak Gaussian potential  $U(\mathbf{k}) = U(0)e^{-\frac{1}{2}(\lambda k)^2}$  corresponding to a small scattering length  $a = -0.1\lambda$ . It is consistent at large distance with the Yukawa potential of Eq. (7). This confirms well-known results for small scattering lengths [24, 25, 35].

On the opposite, one can consider a strong Gaussian potential giving an infinite scattering length. In this case, shown in Fig. 2, the potential is much deeper than the Yukawa potential, and is found numerically to approach a  $-1/r^2$  Efimov potential.

In order to get further insight into the non-perturbative regime, let us now consider the limit of a contact interaction. It corresponds to a Gaussian potential with an infinitesimal range  $\lambda \rightarrow 0$ . Equivalently, the interaction can be taken to be constant in momentum space, i.e.  $U(\mathbf{k}) = g < 0$ , up to some arbitrarily large momentum cutoff  $\Lambda \sim 1/\lambda$ . The scattering length  $a$  of this interaction is given by the relation

$$\frac{2\mu}{4\pi\hbar^2} \frac{1}{a} = \frac{1}{g} + \frac{1}{V} \sum_{|\mathbf{k}| < \Lambda} \frac{1}{\epsilon_k + \varepsilon_k}, \quad (10)$$

which is used to renormalise all final results, i.e. express them in terms of the scattering length  $a$  instead

of  $g$ . Using this interaction in Eqs. (4-5), one encounters the terms  $F_{\mathbf{q}} = g \frac{1}{V} \sum_{\mathbf{p}} u_p \alpha_{\mathbf{q}, \mathbf{p}-\mathbf{q}}$  and  $G_{\mathbf{q}} = g \frac{1}{V} \sum_{\mathbf{p}} v_p \alpha_{\mathbf{q}, \mathbf{p}-\mathbf{q}}$ . Although  $F_{\mathbf{q}}$  remains finite when  $\Lambda \rightarrow \infty$ , since the sum in its expression diverges as  $g^{-1}$  for a fixed value of  $a$ , the term  $G_{\mathbf{q}}$  vanishes, since the sum in its expression does not diverge. In the end, one finds the following equation (see appendix C),

$$\frac{F_{\mathbf{q}}}{T_{\mathbf{q}}(E)} + \frac{1}{V} \sum_{\mathbf{k}} \frac{u_k^2 F_{\mathbf{k}-\mathbf{q}}}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_{\mathbf{q}} - E} = \frac{2n_0}{2\varepsilon_{\mathbf{q}} - E} F_{\mathbf{q}}, \quad (11)$$

where

$$\frac{1}{T_{\mathbf{q}}(E)} = \frac{2\mu}{4\pi\hbar^2} \frac{1}{a} + \frac{1}{V} \sum_{\mathbf{k}} \left( \frac{u_k^2}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_{\mathbf{q}} - E} - \frac{1}{\epsilon_k + \varepsilon_k} \right). \quad (12)$$

As previously, one can find the mediated interaction in the Born-Oppenheimer limit by fixing the distance  $r$  between the two impurities, setting  $\varepsilon_{\mathbf{k}}$  to zero and performing the Fourier transform of these equations. One obtains

$$\frac{2\mu}{4\pi\hbar^2} \frac{1}{a} + \frac{1}{V} \sum_{\mathbf{k}} \left( \frac{u_k^2}{E_k - E} (1 + e^{-i\mathbf{k} \cdot \mathbf{r}}) - \frac{1}{\epsilon_k} \right) = \frac{2n_0}{E}. \quad (13)$$

This equation differs from Eq. (4) of Ref. [33] by its non-zero right-hand side and the coefficient  $u_k^2 \neq 1$ . Let us consider its solution for weak ( $1/a \rightarrow -\infty$ ), unitary ( $1/a = 0$ ), and strong ( $1/a \rightarrow +\infty$ ) boson-impurity interactions.

For small scattering length  $a < 0$  with  $|a| \ll a_B$ , the solution  $E(r)$  of Eq. (13) decays as a weak  $1/r^2$  potential

$$E(r) \xrightarrow{r \rightarrow \infty} E(\infty) - \frac{\hbar^2}{2\mu} \frac{|a|}{\sqrt{2\pi\xi}} \frac{1}{r^2}, \quad (14)$$

where  $E(\infty) \approx 2E_{MF}$ . Interestingly, this potential tail is different from the perturbative result of Eq. (7). However, Fig. 1 shows that the numerical solution of Eq. (13) remains very close to the Yukawa potential of Eq. (7) for  $r \ll \xi$ .

For sufficiently large  $|E| = \frac{\hbar^2}{2\mu} \kappa^2$  and small  $a_B$ , Eq. (13) can be approximated as

$$\frac{1}{a} - \kappa - \frac{4\pi}{\kappa} n_0 a_B + \left( \frac{1}{r} - \frac{4\pi n_0 a_B}{\kappa} \right) e^{-\kappa r} = -\frac{8\pi n_0}{\kappa^2}. \quad (15)$$

It follows that in the large scattering length limit  $a \rightarrow \infty$  and for small  $a_B$ , the mediated interaction has the form

$$E(r) = \begin{cases} -\frac{\hbar^2}{2\mu L^2} \left( 1 + \frac{2}{3} \frac{L}{r} e^{-r/L} \right) & \text{for } r \rightarrow \infty \quad (\text{a}) \\ -\frac{\hbar^2}{2\mu} \frac{W(1)^2}{r^2} & \text{for } r \rightarrow 0 \quad (\text{b}) \end{cases} \quad (16)$$

where  $W(1) \approx 0.567$ . One recognises at short distances the  $1/r^2$  Efimov attraction (in the Born-Oppenheimer limit) between two impurities mediated by a boson. The

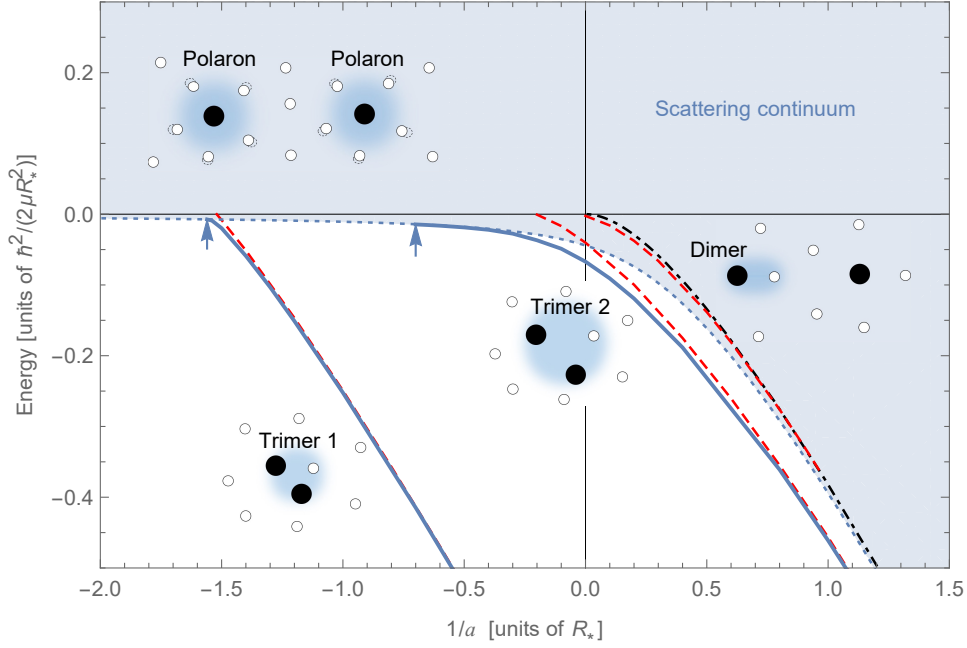


Figure 3. Total energy as a function of inverse scattering length, for the two-channel contact model with  $M/m = 30$ ,  $n_0 = 0.0005R_*^{-3}$ , and  $a_B = R_*$ . The shaded area represents the scattering continuum of the two attractive polarons. Its threshold is closely reproduced by the solution of Eq. (15) for  $r \rightarrow \infty$ , shown as a dotted curve. The solid curves correspond to the bound-state energies  $E$ , solution of Eq. (11). The points where they appear from the polaron scattering threshold are indicated by the vertical arrows. The black dot-dashed curve shows the dimer energy in vacuum, and the red dashed curves correspond to the trimer energies in vacuum.

Efimov attraction can support an infinite number of bound states. However, it is truncated at distances larger than the mean boson spacing  $L = (8\pi n_0)^{-1/3}$ , and asymptotes to the energy  $E(\infty) = -\frac{\hbar^2}{2\mu L^2}$ . As a result, the infinite number of possible trimer states in vacuum is reduced to a finite number, such that only those trimers whose energy is lower or comparable to  $E(\infty)$  survive in the presence of the condensate. This result is qualitatively similar to that of Ref. [33], however here there is no difficulty in calculating  $E(r)$  and its deviation from the Efimov potential occurs at  $r \sim L$  instead of  $r \sim \xi$ . The solution of Eq. (13) as well as its limits Eq. (16) are shown in Fig. 2 and are in good agreement with the Gaussian model calculation for  $r \gg \lambda$ .

Finally, as the boson-impurity interaction is strengthened towards small positive scattering length  $a$ , each polaron is expected to turn into a dimer of energy  $E_d = -\frac{\hbar^2}{2\mu a^2}$ , as each impurity should strongly bind with a nearby boson. However, in the present theory, the asymptotic energy  $E(\infty)$  of the two separated impurities goes to  $E_d$  instead of  $2E_d$ , as can be seen from Eq. (15) for  $r \rightarrow \infty$  and  $1/a \rightarrow +\infty$ . The reason is that the ansatz of Eq. (3) includes only one bosonic excitation, and as a result only one impurity can bind with that excitation. A more quantitative treatment of the scattering threshold of the impurities in the non-perturbative regime would thus require at least two bosonic excitations. The present

theory is therefore quantitative only for  $1/a \lesssim 0$ .

Let us now turn to the energy spectrum of the system. In the contact model, the Efimov attraction exists at infinitely small distances, as seen in Eq. (16), making the problem ill-defined - the so-called Thomas collapse or fall to the centre [27, 36]. Some additional short-range scale is necessary to cure this problem and set the three-body observables. This can be done, for instance, by keeping a finite momentum cutoff  $\Lambda$  for the sum in Eq. (11). Alternatively, one can use a two-channel contact model for the boson-impurity interaction [23]. Such a model corresponds physically to a narrow Feshbach resonance [37]. In this case, the only change in the theory is that the inverse scattering length  $1/a$  in Eq. (12) is replaced by  $\frac{1}{a} - R_* \left[ \left(1 - \frac{\mu^2}{m^2}\right) q^2 - \frac{2\mu E}{\hbar^2} \right]$ , where the range parameter  $R_*$  characterises the narrowness of the Feshbach resonance. Figure 3 represents the exact energy spectrum of the system as a function of  $1/a$  calculated numerically from Eq. (11) with this substitution. A similar spectrum is obtained without this substitution but for a finite momentum cutoff  $\Lambda \sim R_*^{-1}$ .

For any boson-impurity interaction, the spectrum shows a continuum corresponding to scattering states of two attractive polarons. Its threshold, shown by the dotted curve in Fig. 3, corresponds to the asymptotic limit of the mediated interaction, which is closely given by the solution of Eq. (15) for  $r \rightarrow \infty$ . As noted be-

fore, the threshold corresponds to the mean-field energy  $2E_{\text{MF}}$  of two polarons for small  $a < 0$ , and (unphysically) asymptotes to the energy  $E_d$  of a single dimer for small  $a > 0$ . The spectrum also features discrete bound states for sufficiently strong boson-impurity interaction. This is expected since the mediated interaction becomes strong enough to bind the two polarons as it gradually turns from a weak Yukawa potential into a strong Efimov attraction. As the interaction further increases, the bound states (shown as solid curves in Fig. 3) turn into Efimov trimers made of two impurities and one boson. As anticipated from the Born-Oppenheimer potential between the two impurities - see Eq. (16) - only the trimers whose energy is lower than the polaron scattering threshold survive in the presence of the condensate. Near unitarity ( $1/a = 0$ ), the trimer energies are pushed down from the trimer energies in vacuum due the attractive effect of the surrounding bosons, but the binding energies relative to the polaron scattering threshold are smaller than in vacuum. Here, the trimers are therefore weakened by the condensate. However, interestingly, they exist for weaker boson-impurity interactions than in vacuum. The scattering lengths at which the trimers appear (indicated by arrows in Fig. 3) are indeed reduced in magnitude with respect to vacuum. In this sense, the condensate favours the appearance of the trimers. This is especially true when the polaron scattering threshold at unitarity ( $\sim -\hbar^2/(2\mu L^2)$ ) is comparable to the energy of an Efimov trimer in vacuum, as shown by the second trimer in Fig. 3.

Finally, the effect of the boson-boson interaction is considered. It has been assumed that  $a_B \geq 0$  and  $na_B^3 \ll 1$  to ensure the stability of the condensate and the validity of the Bogoliubov treatment. In this range, the trimer energies and the threshold are shifted up almost linearly with  $a_B$ , as shown in Fig. 4. The threshold is slightly less shifted, resulting in a small decrease of the trimer binding energy. Likewise, the boson-boson interactions make the trimers appear from the polaron scattering threshold at a slightly larger boson-impurity attraction strength. Overall, the effect of the boson-boson repulsive interaction is to shift the spectrum up and slightly weaken the bound states.

In summary, a simple variational ansatz has been used to investigate the problem of two impurities in a Bose-Einstein condensate. The ansatz bridges the well-known perturbative regime to the non-perturbative regime, where the Bose-mediated interaction takes the form of the Efimov attraction. It shows that the two polarons formed by the two impurities merge into one or several Efimov trimers for sufficiently strong interaction. The stability of these trimers under the influence of the condensate has also been revealed. Although their binding energy is reduced near unitarity with respect to that of trimers in vacuum, they exist for smaller interaction as the density of the condensate is increased. In a

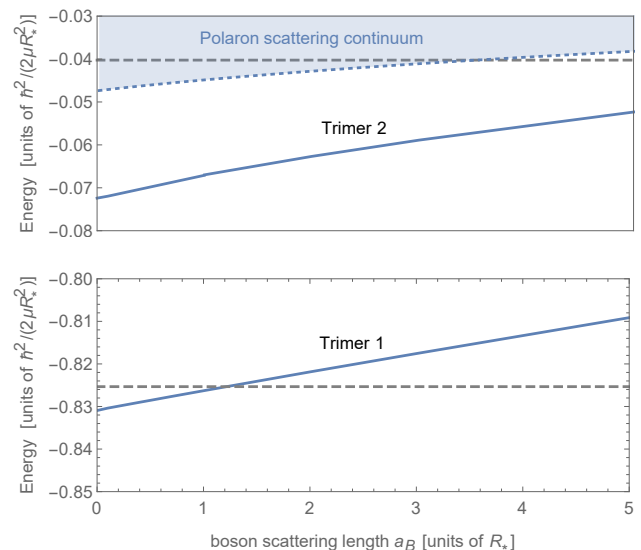


Figure 4. Energy spectrum for  $1/a = 0$  as a function of the boson-boson scattering length  $a_B$ . The density is  $n_0 = 0.0005R_*^{-3}$  and  $M/m = 30$ , as in Fig. 3. The solid curves represent the trimer energies, and the shaded area represents the polaron scattering continuum. The Efimov trimer energies in vacuum are shown in dashed lines for reference.

mixture of resonantly interacting ultra-cold atoms, this would appear as a boson-density-dependent shift of the three-body loss peaks associated with the appearance of Efimov trimers.

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## APPENDIX

### A. Derivation of the Yukawa potential

The Yukawa potential of Eq. (7) is obtained as follows. For sufficiently small scattering length, the term  $E'$  in the denominator of Eq. (6) may be neglected, and the term  $(u_k - v_k)^2 = \epsilon_k/E_k$ . It follows that:

$$E' = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2 \epsilon_k}{E_k^2} (1 + e^{i\mathbf{k} \cdot \mathbf{r}}) \quad (17)$$

$$= -2n_0 \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{\epsilon_k + 2n_0 U_B(0)} (1 + e^{i\mathbf{k} \cdot \mathbf{r}}) \quad (18)$$

Using  $\epsilon_k = \hbar^2 k^2 / (2m)$  and  $U_B(0) = 4\pi\hbar^2 a_B / m$ , one gets

$$E' = \underbrace{-2n_0 \frac{2m}{\hbar^2} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{k^2 + 2\xi^{-2}}}_{E'(\infty)} - \underbrace{2n_0 \frac{2m}{\hbar^2} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{k^2 + 2\xi^{-2}} e^{i\mathbf{k} \cdot \mathbf{r}}}_{E'(r) - E'(\infty)} \quad (19)$$

with  $\xi = 1/\sqrt{8\pi n_0 a_B}$ . The first sum does not depend on  $r$  and converges due to the decay of  $U(k)$  at large  $k$ . In the limit  $\xi \rightarrow \infty$ , one finds

$$E'(\infty) = -2n_0 \frac{2m}{\hbar^2} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{k^2} = -2n_0 \frac{m}{\mu} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{\hbar^2 k^2 / (2\mu)} = \frac{8\pi n_0 \hbar^2}{2\mu} \frac{m}{\mu} a_1$$

where  $a_1 = -\frac{2\mu}{4\pi\hbar^2} \frac{1}{V} \sum_{\mathbf{k}} \frac{U(\mathbf{k})^2}{\epsilon_k + \epsilon_k}$  is the second term of the Born expansion of the scattering length.

The last sum in Eq. (19) goes to zero as  $r \rightarrow \infty$ . Its asymptotic behaviour at large  $r$  may be obtained from the low-momentum contribution in the sum. In this limit,  $U(k)$  may be approximated by  $U(0)$ , i.e.

$$E'(r) - E'(\infty) \xrightarrow{r \rightarrow \infty} -2n_0 \frac{2m}{\hbar^2} U(0)^2 \times \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{k^2 + 2\xi^{-2}} e^{i\mathbf{k} \cdot \mathbf{r}}$$

One recognises in the sum the Fourier transform of  $e^{-\sqrt{2}r/\xi}/(4\pi r)$ , and using the relation  $U(0) = 4\pi\hbar^2 a_0/(2\mu)$ , one finally gets

$$E'(r) - E'(\infty) \xrightarrow{r \rightarrow \infty} -\frac{4\pi\hbar^2 n_0}{\mu^2/m} a_0^2 \frac{e^{-\sqrt{2}r/\xi}}{r},$$

which establishes Eq. (7).

## B. Non-perturbative equation for the Born-Oppenheimer potential

### 1. Derivation

The following details the derivation of Eqs (8) and (9) giving the Born-Oppenheimer potential  $E(r)$  for an arbitrary potential  $U$ .

Setting  $\varepsilon_k = 0$  and  $E' = E - 2nU(0)$  in the general equations Eqs. (4-5) gives

$$-E'\alpha_{\mathbf{q}} + \frac{\sqrt{N_0}}{V} \sum_{\mathbf{k}} U(\mathbf{k})(u_k - v_k)(\alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}}) = 0, \quad (20)$$

$$(E_k - E')\alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} + \mathbf{k})(u_k u_p + v_k v_p)(\alpha_{\mathbf{q},-\mathbf{q}-\mathbf{p}} + \alpha_{\mathbf{q}-\mathbf{p}-\mathbf{k},\mathbf{k}-\mathbf{q}}) + \frac{\sqrt{N_0}}{V} U(\mathbf{k})(u_k - v_k)(\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0, \quad (21)$$

Equation (20) can be written as

$$-E'\alpha_{\mathbf{q}} = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} U(\mathbf{k})(u_k - v_k)\Omega_{\mathbf{k},\mathbf{q}}, \quad (22)$$

with

$$\Omega_{\mathbf{k},\mathbf{q}} = \frac{V}{2\sqrt{N_0}} (\alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}}) \quad (23)$$

Making the change of variable  $\mathbf{q} \rightarrow \mathbf{q} + \mathbf{k}$  in Eq. (21), and then  $\mathbf{k} \rightarrow -\mathbf{k}$ , one obtains

$$(E_k - E')\alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}} + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} - \mathbf{k})(u_k u_p + v_k v_p)(\alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}+\mathbf{k}-\mathbf{p}} + \alpha_{\mathbf{q}-\mathbf{p},-\mathbf{q}}) + \frac{\sqrt{N_0}}{V} U(\mathbf{k})(u_k - v_k)(\alpha_{\mathbf{q}-\mathbf{k}} + \alpha_{\mathbf{q}}) = 0, \quad (24)$$

Summing this equation with Eq. (21), where the change  $\mathbf{p} \rightarrow -\mathbf{p}$  is performed beforehand, and assuming  $U(-\mathbf{k}) = U(\mathbf{k})$ , one gets

$$(E_k - E')(\alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}}) + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} - \mathbf{k})(u_k u_p + v_k v_p)(\alpha_{\mathbf{q},-\mathbf{q}+\mathbf{p}} + \alpha_{\mathbf{q}+\mathbf{p}-\mathbf{k},\mathbf{k}-\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}+\mathbf{k}-\mathbf{p}} + \alpha_{\mathbf{q}-\mathbf{p},-\mathbf{q}}) + 2\frac{\sqrt{N_0}}{V} U(\mathbf{k})(u_k - v_k)(\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0, \quad (25)$$

which can be written as

$$(E_k - E')\Omega_{\mathbf{k},\mathbf{q}} + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} - \mathbf{k})(u_k u_p + v_k v_p)(\Omega_{\mathbf{p},\mathbf{q}} + \Omega_{-\mathbf{p},\mathbf{q}-\mathbf{k}}) + U(\mathbf{k})(u_k - v_k)(\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0. \quad (26)$$

Performing a Fourier transform with respect to  $\mathbf{q}$  of Eq. (22) and (26) gives

$$-E'\alpha(\mathbf{r}) = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} U(\mathbf{k})(u_k - v_k)\Omega_{\mathbf{k}}(\mathbf{r}), \quad (27)$$

$$(E_k - E') \Omega_{\mathbf{k}}(\mathbf{r}) + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} - \mathbf{k}) (u_k u_p + v_k v_p) (\Omega_{\mathbf{p}}(\mathbf{r}) + e^{i\mathbf{k} \cdot \mathbf{r}} \Omega_{-\mathbf{p}}(\mathbf{r})) + U(\mathbf{k}) (u_k - v_k) (\alpha(\mathbf{r}) + e^{i\mathbf{k} \cdot \mathbf{r}} \alpha(\mathbf{r})) = 0. \quad (28)$$

Defining  $A_{\mathbf{k}}(\mathbf{r}) = \Omega_{\mathbf{k}}(\mathbf{r})/\alpha(\mathbf{r})$ , the equations read

$$-E' = -2n_0 \frac{1}{V} \sum_{\mathbf{k}} U(\mathbf{k}) (u_k - v_k) A_{\mathbf{k}}(\mathbf{r}), \quad (29)$$

$$(E_k - E') A_{\mathbf{k}}(\mathbf{r}) + \frac{1}{V} \sum_{\mathbf{p}} U(\mathbf{p} - \mathbf{k}) (u_k u_p + v_k v_p) (A_{\mathbf{p}}(\mathbf{r}) + e^{i\mathbf{k} \cdot \mathbf{r}} A_{-\mathbf{p}}(\mathbf{r})) + U(\mathbf{k}) (u_k - v_k) (1 + e^{i\mathbf{k} \cdot \mathbf{r}}) = 0. \quad (30)$$

Considering the even parity sector  $A_{-\mathbf{p}}(\mathbf{r}) = A_{\mathbf{p}}(\mathbf{r})$  corresponding to the bonding solution, one finally obtains Eqs. (8) and (9).

## 2. Numerical solution

The large volume limit is obtained by replacing the sums  $\frac{1}{V} \sum_{\mathbf{p}}$  by integrals  $(2\pi)^{-3} \int d^3\mathbf{p}$ . The equations can be further simplified for an isotropic potential  $U(\mathbf{k}) = U(k)$  and assuming the isotropic form  $A_{\mathbf{p}}(\mathbf{r}) = A_p(r)$ . Integrating over the direction of  $\mathbf{p}$ , then over the direction of  $\mathbf{k}$  yields:

$$-E' = -2n_0 \frac{1}{2\pi^2} \int_0^\infty k^2 dk U(k) (u_k - v_k) A_k(r), \quad (31)$$

$$(E_k - E') A_k(r) + \frac{1}{2\pi^2} \int_0^\infty p^2 dp U(p, k) (u_k u_p + v_k v_p) \left(1 + \frac{\sin kr}{kr}\right) A_p(r) + U(k) (u_k - v_k) \left(1 + \frac{\sin kr}{kr}\right) = 0, \quad (32)$$

where

$$U(p, k) = \frac{1}{4\pi} \int d\Omega_{\mathbf{p}} U(\mathbf{p} - \mathbf{k}) \quad (33)$$

In the case of the Gaussian potential  $U(\mathbf{p}) = U(0)e^{-\frac{1}{2}(\lambda p)^2}$ , one has

$$U(p, k) = U(0) \frac{e^{-\frac{1}{2}\lambda^2(p+k)^2} (e^{2kp\lambda^2} - 1)}{2kp\lambda^2} \quad (34)$$

The equations (31) and (32) can be solved numerically by discretising the momenta  $k$  and  $p$ , leading to a matrix problem that can be solved using standard linear algebra routines.

## C. Equation and solution for the contact model

### 1. Derivation

The following provides the derivation of Eq. (11).

Starting from the general equations Eqs. (4-5), one performs the changes  $\alpha_{\mathbf{q}-\mathbf{k},-\mathbf{q}} = \alpha_{-\mathbf{q},\mathbf{q}-\mathbf{k}}$  in Eq. (4) and  $\alpha_{\mathbf{q}-\mathbf{p}-\mathbf{k},\mathbf{k}-\mathbf{q}} = \alpha_{\mathbf{k}-\mathbf{q},\mathbf{q}-\mathbf{p}-\mathbf{k}}$  in Eq. (5), using the bosonic exchange symmetry  $\alpha_{\mathbf{q},\mathbf{q}'} = \alpha_{\mathbf{q}',\mathbf{q}}$ . Then, one sets the potential  $U(\mathbf{k}) = g$  for  $k < \Lambda$ ,  $U(\mathbf{k}) = 0$  for  $k \geq \Lambda$ . This yields

$$(2\varepsilon_q - E') \alpha_{\mathbf{q}} + g \frac{\sqrt{N_0}}{V} \sum_{\mathbf{k}}^{k < \Lambda} (u_k - v_k) (\alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + \alpha_{-\mathbf{q},\mathbf{q}-\mathbf{k}}) = 0 \quad (35)$$

$$(E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E') \alpha_{\mathbf{q},\mathbf{k}-\mathbf{q}} + g \frac{1}{V} \sum_{\mathbf{p}}^{|\mathbf{p}+\mathbf{k}| < \Lambda} (u_k u_p + v_k v_p) (\alpha_{\mathbf{q},-\mathbf{q}-\mathbf{p}} + \alpha_{\mathbf{k}-\mathbf{q},\mathbf{q}-\mathbf{p}-\mathbf{k}}) + g \frac{\sqrt{N_0}}{V} (u_k - v_k) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0, \quad (36)$$



In the first equation, one can change the term  $\alpha_{-\mathbf{q}, \mathbf{q}-\mathbf{k}}$  into  $\alpha_{-\mathbf{q}, \mathbf{k}+\mathbf{q}}$  by performing the change of variable  $\mathbf{k} \rightarrow -\mathbf{k}$ . The equation then reads,

$$(2\varepsilon_q - E') \alpha_{\mathbf{q}} + \sqrt{N_0} (F_{\mathbf{q}} + F_{-\mathbf{q}} - G_{\mathbf{q}} - G_{-\mathbf{q}}) = 0 \quad (37)$$

where the terms  $F_{\mathbf{q}}$  and  $G_{\mathbf{q}}$  are defined by

$$F_{\mathbf{q}} = g \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} u_k \alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}} \quad (38)$$

$$G_{\mathbf{q}} = g \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} v_k \alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}} \quad (39)$$

Next, the change of variable  $\mathbf{p} \rightarrow -\mathbf{p}$  is performed in Eq. (36). For sufficiently large  $\Lambda$ , the sum  $\sum_{\mathbf{p}}^{|\mathbf{p}-\mathbf{k}| < \Lambda}$  can be approximated by  $\sum_{\mathbf{p}}^{p < \Lambda}$ , so that Eq. (36) can be expressed in terms of  $F$  and  $G$ :

$$(E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E') \alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}} + u_k (F_{\mathbf{q}} + F_{\mathbf{k}-\mathbf{q}}) + v_k (G_{\mathbf{q}} + G_{\mathbf{k}-\mathbf{q}}) + g \frac{\sqrt{N_0}}{V} (u_k - v_k) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}}) = 0 \quad (40)$$

Using this equation to express  $\alpha_{\mathbf{q}, \mathbf{k}-\mathbf{q}}$  in Eqs. (38) and (39), one finds

$$\frac{1}{g} F_{\mathbf{q}} = -\frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} u_k \frac{u_k (F_{\mathbf{q}} + F_{\mathbf{k}-\mathbf{q}}) + v_k (G_{\mathbf{q}} + G_{\mathbf{k}-\mathbf{q}}) + g \frac{\sqrt{N_0}}{V} (u_k - v_k) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}})}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \quad (41)$$

$$\frac{1}{g} G_{\mathbf{q}} = -\frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} v_k \frac{u_k (F_{\mathbf{q}} + F_{\mathbf{k}-\mathbf{q}}) + v_k (G_{\mathbf{q}} + G_{\mathbf{k}-\mathbf{q}}) + g \frac{\sqrt{N_0}}{V} (u_k - v_k) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}})}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \quad (42)$$

Owing to the renormalisation relation Eq. (10), for a fixed scattering length  $a$ , the term  $1/g$  in the left-hand side of Eqs. (41) and (42) diverges as  $\Lambda$  for very large  $\Lambda$ . In Eq. (41), this divergence in the left-hand side is cancelled by another divergent term in the right-hand side. In contrast, in Eq. (42), the right-hand side does not diverge for large  $\Lambda$ . Indeed, for large  $\mathbf{k}$ , the denominator in Eq. (42) is  $\sim k^2$ , and the numerator involves the terms  $v_k u_k \sim k^{-2}$  and  $v_k^2 \sim k^{-4}$ . The term in the sum of Eq. (42) thus decay as  $k^{-4}$  or faster, and the sum is therefore convergent. One concludes that  $G$  may be neglected for sufficiently large  $\Lambda$ .

There only remain two equations, Eq. (37) and (41), which for  $G = 0$  read

$$(2\varepsilon_q - E') \alpha_{\mathbf{q}} + \sqrt{N_0} (F_{\mathbf{q}} + F_{-\mathbf{q}}) = 0 \quad (43)$$

$$\frac{1}{g} F_{\mathbf{q}} = -\frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k^2 (F_{\mathbf{q}} + F_{\mathbf{k}-\mathbf{q}}) + g \frac{\sqrt{N_0}}{V} u_k (u_k - v_k) (\alpha_{\mathbf{q}} + \alpha_{\mathbf{q}-\mathbf{k}})}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \quad (44)$$

One can further rewrite Eq. (44) as

$$\begin{aligned} \left( \frac{1}{g} + \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k^2}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \right) F_{\mathbf{q}} = & -\frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k^2}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} F_{\mathbf{k}-\mathbf{q}} \\ & - g \frac{\sqrt{N_0}}{V} \left[ \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k (u_k - v_k)}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \right] \alpha_{\mathbf{q}} \\ & - g \frac{\sqrt{N_0}}{V} \left[ \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k (u_k - v_k) \alpha_{\mathbf{q}-\mathbf{k}}}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} \right] \end{aligned} \quad (45)$$

The sum in the last term is convergent, since for large  $\mathbf{k}$ ,  $u_k (u_k - v_k) \sim 1$  and  $\alpha_{\mathbf{q}-\mathbf{k}} \lesssim k^{-2}$  according to Eq. (43). Since it is multiplied by the vanishing factor  $g$ , it can therefore be neglected. On the other hand, the sum in the

second term of Eq. (45) diverges as  $1/g$  - as seen from Eq. (10). It therefore cancels the factor  $g$  for large enough  $\Lambda$ . Finally, the divergence of the sum in the left-hand side of Eq. (45) is cancelled by the term  $1/g$ , which can be done explicitly by using the renormalisation relation Eq. (10). Finally, Eq. (45) simplifies to

$$\frac{1}{T_q(E')} F_q + \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \frac{u_k^2}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} F_{\mathbf{k}-\mathbf{q}} = -g \frac{\sqrt{N_0}}{V} \alpha_q \quad (46)$$

with

$$\frac{1}{T_q(E')} = \frac{2\mu}{4\pi\hbar^2} \frac{1}{a} + \frac{1}{V} \sum_{\mathbf{k}}^{k < \Lambda} \left( \frac{u_p^2}{E_k + \varepsilon_{|\mathbf{k}-\mathbf{q}|} + \varepsilon_q - E'} - \frac{1}{\epsilon_k + \varepsilon_k} \right). \quad (47)$$

The energy  $E' = E - 2ng$  may be replaced by  $E$  since  $g$  vanishes for large  $\Lambda$ . As a last step, one can perform the changes  $\mathbf{q} \rightarrow -\mathbf{q}$  and  $\mathbf{k} \rightarrow -\mathbf{k}$  in Eq. (46), and observe that  $F_{-\mathbf{q}}$  satisfies the same equation as  $F_{\mathbf{q}}$ . One can thus set  $F_{-\mathbf{q}} = F_{\mathbf{q}}$  in Eq. (43). Combining this equation with Eq. (46) finally yields Eqs. (11).

It is worthwhile to note that the same equation can be obtained in a different way from a two-channel contact model, whose range parameter is set to zero.

## 2. Numerical solution

In the  $s$ -wave channel ( $F_{\mathbf{q}} = F_q$ ), where the Efimov attraction takes place, the equation (11) simplifies as follows,

$$\left( \frac{1}{T_q(E)} - \frac{2n_0}{2\varepsilon_q - E} \right) F_q + \frac{1}{V} \sum_{\mathbf{k}} \frac{u_{|\mathbf{k}+\mathbf{q}|}^2}{E_{|\mathbf{k}+\mathbf{q}|} + \varepsilon_k + \varepsilon_q - E} F_k = 0. \quad (48)$$

Making the substitution  $\sum_{\mathbf{k}} \approx (2\pi)^{-3} \int d^3\mathbf{k}$  and using the explicit forms of  $T_k(E)$ ,  $E_k$  and  $\varepsilon_k$  gives

$$M_q(z) F_q + \int_0^\infty dk M_{qk}(z) F_k = 0. \quad (49)$$

where

$$M_q(z) = \frac{1}{a} + \frac{2}{\pi} \int_0^\Lambda dk \left( \frac{M}{\mu} \frac{k u_k^2}{q} \arctan \frac{kq}{\frac{M}{m} \sqrt{k^2(k^2 + 2\xi^{-2})} + (k^2 + 2q^2) - \frac{M}{\mu} z} - 1 \right) - \frac{8\pi n_0}{\frac{2\mu}{M} q^2 - z}, \quad (50)$$

$$M_{qk}(z) = \frac{1}{\pi} \frac{k}{q} \int_{|k-q|}^{|k+q|} \frac{p u_p^2}{\frac{\mu}{m} p \sqrt{p^2 + 2\xi^{-2}} + \frac{\mu}{M} (k^2 + q^2) - z} dp, \quad (51)$$

and  $z = \frac{2\mu}{\hbar^2} E$ . Equation (49) can be solved as a matrix problem by discretising the momenta  $q$  and  $k$  on a grid. The eigenvalues can be found by standard linear algebra routines, and the energy levels are obtained by finding the values of  $z$  which make one of the eigenvalues equal to zero.